Quaternionic plurisubharmonic functions and their applications to convexity.

Semyon Alesker *
Department of Mathematics, Tel Aviv University, Ramat Aviv
69978 Tel Aviv, Israel
e-mail: semyon@post.tau.ac.il

Dedicated to Professor Victor Abramovich Zalgaller in occasion of his 85-th birthday.

Abstract

The goal of this article is to present a survey of the recent theory of plurisubharmonic functions of quaternionic variables, and its applications to theory of valuations on convex sets and HKT-geometry (HyperKähler with Torsion). The exposition follows the articles [4], [5], [7] by the author and [8] by M. Verbitsky and the author.

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0 Introduction.

The goal of this article is to present a survey of the recent theory of plurisubharmonic functions of quaternionic variables, and its applications to theory of valuations on convex sets and HKT-geometry (HyperKähler with Torsion). The exposition follows the articles [4], [5], [7] by the author and [8] by M. Verbitsky and the author.

We will denote by $\mathbb{H}$ the (non-commutative) field of quaternions. The notion of quaternionic plurisubharmonic function on the flat space $\mathbb{H}^n$ was introduced by the author in [4] and independently by G. Henkin [24]. This notion is a quaternionic analogue of convex functions on $\mathbb{R}^n$ and complex plurisubharmonic functions on $\mathbb{C}^n$, see Definition 3.1 below. On one hand, this class of functions obeys many analytical properties analogous to those of convex and complex plurisubharmonic functions. On the other hand, these properties reflect rather different geometric structures behind them. This will be illustrated below on applications to convexity and HKT-geometry.

Let us start with some analytical properties of quaternionic plurisubharmonic functions. The author has proved in [4] a quaternionic analogue of the A.D. Aleksandrov [2] and Chern-Levine-Nirenberg [19] theorems (see Theorems 3.4, 3.6 in Section 3 below). It is worthwhile to remind these classical results now. The Aleksandrov theorem says that if a sequence $\{f_N\}$ of *convex* functions converges uniformly on compact subsets to a function f, then f is convex and

$$\det\left(\frac{\partial^2 f_N}{\partial x_i \partial x_j}\right) \xrightarrow{w} \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

weakly in the sense of measures (note that the expression $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ for a convex function u is understood in a generalized sense as explained in the quaternionic situation in Theorem 3.4 of Section 3 below).

The Chern-Levine-Nirenberg theorem (in fact, in a slightly weaker form) says that if a sequence $\{f_N\}$ of *continuous complex plurisubharmonic* functions converges uniformly on compact subsets to a function f, then f is continuous complex plurisubharmonic and

$$\det\left(\frac{\partial^2 f_N}{\partial z_i \partial \bar{z}_j}\right) \xrightarrow{w} \det\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)$$

weakly in the sense of measures (again the expressions $\det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)$ is understood in a generalized sense).

The statement of the quaternionic analogue of the above theorems requires analogues of complex operators $\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}$ and the notion of determinant of quaternionic matrices. The former notion is called sometimes Dirac operators $\frac{\partial}{\partial \bar{q}}, \frac{\partial}{\partial q}$; it is discussed in Section 2. The latter notion of quaternionic determinants is discussed in Section 1 where we discuss in detail the Moore determinant of hyperhermitian (= quaternionic hermitian) matrices. This quaternionic result is used in applications to theory of valuations on convex sets (Theorem 4.2 in Section 4).

Another important for applications in valuation theory result is Theorem 3.7 (proved in [7]) which is a quaternionic version of Błocki's theorem [13] for complex plurisubharmonic functions.

In Section 4 we discuss in more details applications of the above results to this theory of valuations on convex sets. Let us remind basic notions of this theory referring for further information to the surveys by McMullen [31] and McMullen and Schneider [32]. Let V be a finite dimensional real vector space. Let $\mathcal{K}(V)$ denote the class of all non-empty convex compact subsets of V.

0.1 Definition. (1) A function $\phi : \mathcal{K}(V) \to \mathbb{C}$ is called a valuation if for any $K_1,\ K_2 \in \mathcal{K}(V)$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

(2) A valuation ϕ is called continuous if it is continuous with respect the Hausdorff metric on $\mathcal{K}(V)$.

Recall that the Hausdorff metric d_H on $\mathcal{K}(V)$ depends on a choice of a Euclidean metric on V and it is defined as follows: $d_H(A, B) := \inf\{\varepsilon > 0 | A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\}$ where $(U)_\varepsilon$ denotes the ε-neighborhood of a set U. Then $\mathcal{K}(V)$ becomes a locally compact space, and the topology on $\mathcal{K}(V)$ induced by the Hausdorff metric does not depend on a choice of the Euclidean metric on V.

The theory of valuations has numerous applications in convexity and integral geometry (see e.g. [6], [23], [28], [38]). Let us remind some basic examples of translation invariant continuous valuations.

0.2 Example. (1) A Lebesgue measure *vol* on V is a translation invariant continuous valuation.

(2) The Euler characteristic χ is a translation invariant continuous valuation. (Recall that $\chi(K) = 1$ for any $K \in \mathcal{K}(V)$.)

(3) Denote $m := \dim V$. Fix $k = 1, \dots, m$. Fix $A_1, \dots, A_{m-k} \in \mathcal{K}(V)$. Then the mixed volume

$$K \mapsto V(K[k], A_1, \dots, A_{m-k})$$

is a translation invariant continuous valuations. (For the notion of mixed volume and its properties see e.g. the books by Burago-Zalgaller [14] and Schneider [38].)

It was conjectured by P. McMullen [30] and proved by the author [3] that the linear combinations of the mixed volumes as in Example 0.2 (3) above are dense in the space of all translation invariant continuous valuations in the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$.

Nevertheless there are other than mixed volumes non-trivial constructions of translation invariant continuous valuations. In this survey we will discuss two of them. They are closely related to each other since they are based on the theory of complex and, respectively, quaternionic plurisubharmonic functions. Their relation to the mixed volume construction is not straightforward. The possibility of approximating of these examples by the mixed volumes follows from the solution of the McMullen's conjecture.

Let us agree on the notation. For $K \in \mathcal{K}(V)$ one denotes by $h_K \colon V^* \to \mathbb{R}$ the supporting functional of K. Recall that

$$h_K(y) = \sup\{y(x) |\, y \in K\}.$$

Let us describe the construction of valuations using complex plurisubharmonic functions. Let us denote by $\Omega^{p,p}$ the vector bundle over $\mathbb{C}^n$ of (p,p)-forms. Let us denote by $C_c(\mathbb{C}^n,\Omega^{p,p})$ the space of continuous compactly supported forms of type (p,p) on $\mathbb{C}^n$.

0.3 Theorem ([7], Theorem 4.1.3). *Fix $k=1,\dots,n$. Fix $\psi\in C_0(\mathbb{C}^n,\Omega^{n-k,n-k})$. Then $K\mapsto\int_{\mathbb{C}^n}(dd^ch_K)^k\wedge\psi$ defines a continuous translation invariant valuation on $\mathcal{K}(\mathbb{C}^n)$.*

This result in the above generality was proved by the author in [7] using some known properties of complex plurisubharmonic functions. This theorem contains two non-trivial parts: the continuity and the valuation property of the above functional. The former is a consequence of the Chern-Levine-Nirenberg theorem [19], and the latter is a consequence of the Błocki formula [13]. Note that the expressions of the form as in Theorem 4.1 were considered first in the context of convexity probably by Kazarnovskiĭ[26], [27]. In [7] we have obtained quaternionic version of the above construction; it is discussed in Section 4, Theorem 4.2.

In Section 5 we discuss two theorems on the Dirichlet problem for quaternionic Monge-Ampère equations obtained by the author in [5]. They also have classical real and complex analogues; we refer to Section 5 for references.

In Section 6 we describe generalizations of some of the definitions and results on quaternionic plurisubharmonic functions to so called hypercomplex manifolds due to M. Verbitsky and the author [8]. This class of manifolds contains, for instance, the flat spaces $\mathbb{H}^n$ and hyperKähler manifolds. On such manifolds one defines quaternionic plurisubharmonic functions and proves for them a version of the Aleksandrov and Chern-Levine-Nirenberg theorems. Next it turns out that C^∞-smooth strictly plurisubharmonic functions on hypercomplex manifolds admit a geometric interpretation as (local) potentials of HKT-metrics; it was also obtained in [8]. Roughly put, an HKT-metrics on a hypercomplex manifold is an $SU(2)$-invariant Riemannian metric satisfying certain first order differential equations. These metrics are analogous to Kähler metrics on complex manifolds. The above mentioned interpretation of quaternionic plurisubharmonic functions is analogous to the well known interpretation of C^∞-smooth complex strictly plurisubharmonic functions on complex manifolds as (local) potentials of Kähler metrics. Let us recall more explicitly the last fact. Let f be a C^∞-smooth complex strictly plurisubharmonic function on a complex manifold M. Let us fix on M local complex coordinates. Then the matrix $g:=\left(\frac{\partial^2 f}{\partial z_i\partial\bar z_j}\right)$ defines a Kähler metric on M. Vice versa, for any Kähler metric g on M, every point $z\in M$ has a neighborhood such that in this neighborhood $g=\left(\frac{\partial^2 f}{\partial z_i\partial\bar z_j}\right)$ for a C^∞-smooth complex strictly plurisubharmonic function f.

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1 Quaternionic linear algebra.

It is known that some of the standard results of linear algebra over commutative fields can be generalized to general non-commutative fields, e.g. theory of dimension and basis of vector spaces. However over non-commutative fields there is no notion of determinant of matrices

which would be as good as in the commutative case. There is a notion of the Diedonné determinant (see e.g. [9]) which is good for some applications (e.g. [4]). Gelfand and Retakh have developed a theory of quasi-determinants over non-commutative fields which generalizes in a sense probably all the known theories of non-commutative determinants (in particular the Dieudonné, super-, quantum- determinants). We will not discuss this theory here, and refer to the survey [20].

Nevertheless over the field $\mathbb{H}$ of quaternions there is a notion of *Moore determinant* on the class of *hyperhermitian* matrices discussed below. Hyperhermitian quaternionic matrices are analogous to real symmetric and complex hermitian matrices. The properties of the Moore determinant on this class are very similar to the properties of the usual determinant. It seems that any general identity or inequality which is true for the determinant of real symmetric or complex hermitian matrices should be true for the Moore determinants of quaternionic hyperhermitian matrices. Among examples of such results one can mention Sylvester criterion of positive definiteness and Aleksandrov inequalities for mixed determinants discussed below. For more information on quaternionic determinants see the survey [10] and references therein; for the relation of the Moore determinant to the Gelfand-Retakh quasideterminants see [21].

In the rest of this section we discuss in more detail the notion of hyperhermitian matrices and their Moore determinant.

1.1 Definition. A *hyperhermitian semilinear form* on V is a map $a : V \times V \longrightarrow \mathbb{H}$ satisfying the following properties:

(a) a is additive with respect to each argument;

(b) $a(x, y \cdot q) = a(x, y) \cdot q$ for any $x, y \in V$ and any $q \in \mathbb{H}$;

(c) $a(x, y) = \overline{a(y, x)}$ where $\bar{q}$ denotes the usual conjugation of a quaternion q

1.2 Definition. A quaternionic $(n \times n)$-matrix $A = (a_{ij})_{i,j=1}^n$ is called *hyperhermitian* if $a_{ij} = \bar{a}_{ji}$.

1.3 Example. Let $V = \mathbb{H}^n$ be the standard coordinate space considered as right vector space over $\mathbb{H}$. Fix a *hyperhermitian* $(n \times n)$-matrix $(a_{ij})_{i,j=1}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define

$$a(x, y) = \sum_{i,j} \bar{x}_i a_{ij} y_j$$

(note the order of the terms!). Then a defines hyperhermitian semilinear form on V.

In general one has the following standard claims.

1.4 Claim. *Fix a basis in a finite dimensional right quaternionic vector space V. Then there is a natural bijection between hyperhermitian semilinear forms on V and $(n \times n)$-hyperhermitian matrices.*

This bijection is in fact described in previous Example 1.3.

1.5 Claim. *Let A be the matrix of a given hyperhermitian form in a given basis. Let C be transition matrix from this basis to another one. Then the matrix A' of the given form in the new basis is equal to C^*AC.*

1.6 Definition. A hyperhermitian semilinear form a is called *positive definite* if $a(x,x)>0$ for any non-zero vector x.

Let us fix on our quaternionic right vector space V a positive definite hyperhermitian form $(\cdot,\cdot)$. The space with fixed such a form will be called *hyperhermitian* space.

For any quaternionic linear operator $\phi: V\to V$ in hyperhermitian space V one can define the adjoint operator $\phi^*: V\to V$ in the usual way, i.e. $(\phi x,y)=(x,\phi^* y)$ for any $x,y\in V$. Then if one fixes an orthonormal basis in the space V then the operator ϕ is selfadjoint if and only if its matrix in this basis is hyperhermitian.

1.7 Claim. *For any selfadjoint operator in a hyperhermitian space there exists an orthonormal basis such that its matrix in this basis is diagonal and real.*

Now we are going to define the Moore determinant of hyperhermitian matrices. The definition below is different from the original one due to Moore [33] but equivalent to it.

First note that every hyperhermitian $(n\times n)$- matrix A defines a hyperhermitian semilinear form on the coordinate space $\mathbb{H}^n$ as explained in Example 1.3. It also can be considered as a *symmetric* bilinear form on $\mathbb{R}^{4n}$ (which is the realization of $\mathbb{H}^n$). Let us denote its $(4n\times 4n)$- matrix by ${}^{\mathbf{R}}A$. Let us consider the entries of A as formal variables (each quaternionic entry corresponds to four commuting real variables). Then $det({}^{\mathbf{R}}A)$ is a homogeneous polynomial of degree $4n$ in $n(2n-1)$ real variables. Let us denote by Id the identity matrix. One has the following result which was rediscovered several times (see [10] for references).

1.8 Theorem. *There exists a polynomial P defined on the space of all hyperhermitian $(n\times n)$-matrices such that for any hyperhermitian $(n\times n)$-matrix A one has $det({}^{\mathbf{R}}A)=P^4(A)$ and $P(Id)=1$. P is defined uniquely by these two properties. Furthermore P is homogeneous of degree n and has integer coefficients.*

Thus for any hyperhermitian matrix A the value $P(A)$ is a real number, and it is called the *Moore determinant* of the matrix A. The explicit formula for the Moore determinant was given by Moore [33] (see also [10] and Theorem 1.16 below). From now on the Moore determinant of a matrix A will be denoted by $detA$. This notation should not cause any confusion with the usual determinant of real or complex matrices due to part (i) of the next theorem which is probably a folklore.

1.9 Theorem. *(i) The Moore determinant of any complex hermitian matrix considered as quaternionic hyperhermitian matrix is equal to its usual determinant.*

(ii) For any hyperhermitian matrix A and any quaternionic matrix C

$$det(C^*AC)=detA\cdot det(C^*C).$$

1.10 Example. (a) Let $A=diag(\lambda_1,\dots,\lambda_n)$ be a diagonal matrix with real λ_i's. Then A is hyperhermitian and the Moore determinant $detA=\prod_i\lambda_i$.

(b) A general hyperhermitian (2×2)-matrix A has the form

$$A=\begin{bmatrix} a & q\\ \bar q & b\end{bmatrix},$$

where $a,b\in\mathbb{R}$, $q\in\mathbb{H}$. Then $detA=ab-q\bar q(=ab-\bar q q)$.

1.11 Claim. *Let A be a non-negative (resp. positive) definite hyperhermitian matrix. Then $detA \geq 0$ (resp. $detA > 0$).*

The following theorem is a quaternionic generalization of the standard Sylvester criterion.

1.12 Theorem (Sylvester criterion, [4]). *A hyperhermitian $(n \times n)$- matrix A is positive definite if and only if the Moore determinants of all the left upper minors of A are positive.*

Let us define now the mixed discriminant of hyperhermitian matrices in analogy with the case of real symmetric matrices studied by A.D. Aleksandrov [1].

1.13 Definition. Let $A_1, \dots, A_n$ be hyperhermitian $(n \times n)$- matrices. Consider the homogeneous polynomial in real variables $\lambda_1, \dots, \lambda_n$ of degree n equal to $det(\lambda_1 A_1 + \dots + \lambda_n A_n)$. The coefficient of the monomial $\lambda_1 \cdot \dots \cdot \lambda_n$ divided by $n!$ is called the *mixed discriminant* of the matrices $A_1, \dots, A_n$, and it is denoted by $\det(A_1, \dots, A_n)$.

Note that the mixed discriminant is symmetric with respect to all variables, and linear with respect to each of them; recall that the linearity with respect to say the first argument means that

$$\det(\lambda A_1' + \mu A_1'', A_2, \dots, A_n) = \lambda \cdot \det(A_1', A_2, \dots, A_n) + \mu \cdot \det(A_1'', A_2, \dots, A_n)$$

for any *real* λ, μ. Note also that $\det(A, \dots, A) = detA$. One has the following generalization of Aleksandrov's inequalities for mixed discriminants [1].

1.14 Theorem. *(i) The mixed discriminant of positive (resp. non-negative) definite matrices is positive (resp. non-negative).*

(ii) Fix positive definite hyperhermitian $(n \times n)$- matrices $A_1, \dots, A_{n-2}$. On the real linear space of hyperhermitian $(n \times n)$- matrices consider the bilinear form

$$B(X, Y) := \det(X, Y, A_1, \dots, A_{n-2}).$$

Then B is non-degenerate quadratic form, and its signature has one plus and the rest are minuses.

1.15 Corollary (Aleksandrov inequality, [4]). *Let $A_1, \dots, A_{n-1}$ be positive definite hyperhermitian $(n \times n)$- matrices. Then for any hyperhermitian matrix X*

$$\det(A_1, \dots, A_{n-1}, X)^2 \geq \det(A_1, \dots, A_{n-1}, A_{n-1}) \cdot \det(A_1, \dots, A_{n-2}, X, X),$$

and the equality is satisfied if and only if the matrix X is proportional to A_{n-1}.

Finally let us give an explicit formula for the Moore determinant (which was the original definition by Moore [33]). Let $A = (a_{i,j})_{i,j=1}^n$ be a hyperhermitian $(n \times n)$-matrix. Let σ be a permutation of $\{1, \dots, n\}$. Write σ as a product of disjoint cycles such that each cycle starts with the smallest number. Since disjoint cycles commute we may write

$$\sigma = (k_{11} \dots k_{1j_1})(k_{21} \dots k_{2j_2}) \dots (k_{m1} \dots k_{mj_m})$$

where for each i we have $k_{i1} < k_{ij}$ for all $j > 1$, and $k_{11} > k_{21} > \dots > k_{m1}$. This expression is unique. Let $sgn(\sigma)$ is the parity of σ. For the next result we refer to [10] and references therein.

1.16 Theorem. *The Moore determinant of A is equal to*

$$\det A = \sum_{\sigma} sgn(\sigma) a_{k_{11},k_{12}} \dots a_{k_{1j_1},k_{11}} a_{k_{21},k_{22}} \dots a_{k_{mj_m},k_{m1}}$$

where the sum runs over all permutations σ.

2 Dirac operators.

We will write a quaternion $q \in \mathbb{H}$ in the standard form

$$q = t + x \cdot i + y \cdot j + z \cdot k,$$

where t, x, y, z are real numbers, and i, j, k satisfy the usual relations

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

The Dirac operator $\frac{\partial}{\partial \bar q}$ is defined as follows. For any $\mathbb{H}$-valued function F

$$\frac{\partial}{\partial \bar q} F := \frac{\partial F}{\partial t} + i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}.$$

Let us also define the operator $\frac{\partial}{\partial q}$:

$$\frac{\partial}{\partial q} F := \overline{\frac{\partial}{\partial \bar q} \bar F} = \frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} i - \frac{\partial F}{\partial y} j - \frac{\partial F}{\partial z} k.$$

In the case of several quaternionic variables, it is easy to see that the operators $\frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial \bar q_j}$ commute:

$$[\frac{\partial}{\partial q_i}, \frac{\partial}{\partial \bar q_j}] = 0. \tag{1}$$

2.1 Proposition ([4]). *(i) Let $f : \mathbb{H}^n \to \mathbb{H}$ be a smooth function. Then for any $\mathbb{H}$-linear transformation A of $\mathbb{H}^n$ (as a right $\mathbb{H}$-vector space) one has the identities*

$$\left(\frac{\partial^2 f(Aq)}{\partial \bar q_i \partial q_j}\right) = A^* \left(\frac{\partial^2 f}{\partial \bar q_i \partial q_j}(Aq)\right) A.$$

(ii) If, in addition, f is real valued then for any $\mathbb{H}$-linear transformation A of $\mathbb{H}^n$ and any quaternion a with $|a| = 1$

$$\left(\frac{\partial^2 f(A(q \cdot a))}{\partial \bar q_i \partial q_j}\right) = A^* \left(\frac{\partial^2 f}{\partial \bar q_i \partial q_j}(A(q \cdot a))\right) A.$$

3 Plurisubharmonic functions of quaternionic variables.

First we introduce the class of quaternionic plurisubharmonic functions on $\mathbb{H}^n$ following [4]. Note that this notion was also introduced independently by G. Henkin [24]. Let Ω be an open subset of $\mathbb{H}^n$.

3.1 Definition. A real valued function $u : \Omega \to \mathbb{R}$ is called quaternionic plurisubharmonic if it is upper semi-continuous and its restriction to any right *quaternionic* line is subharmonic.

Recall that upper semi-continuity means that $u(x_0) \geq \limsup_{x \to x_0} u(x)$ for any $x_0 \in \Omega$. We will denote by $P(\Omega)$ the class of plurisubharmonic functions in the open subset Ω.

Also we will call a C^2-smooth function $u : \Omega \to \mathbb{R}$ to be *strictly plurisubharmonic* if its restriction to any right quaternionic line is strictly subharmonic (i.e. the Laplacian is strictly positive).

Before we state the next proposition let us observe that if a smooth function f is real valued then the matrix $(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j})(q)$ is hyperhermitian.

3.2 Proposition ([4], Prop. 2.1.6). *A real valued twice continuously differentiable function f on an open subset $\Omega \subset \mathbb{H}^n$ is quaternionic plurisubharmonic (reps. strictly plurisubharmonic) if and only if at every point $q \in \Omega$ the matrix $(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j})(q)$ is non-negative definite (resp. positive definite).*

3.3 Remark. Proposition 3.2 is completely analogous to characterization of smooth convex function as functions with non-negative definite Hessian $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$, and smooth complex plurisubharmonic functions as functions with non-negative definite complex Hessian $\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)$.

3.4 Theorem ([4]). *For any function $u \in C(\Omega) \cap P(\Omega)$ one can uniquely define a non-negative measure $\det(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j})$ which is uniquely characterized by the following two properties:*
(1) if $u \in C^2(\Omega)$ then it has the obvious meaning;
(2) if $u_N \to u$ uniformly on compact subsets in Ω, and u_N, $u \in C(\Omega) \cap P(\Omega)$, then

$$\det(\frac{\partial^2 u_N}{\partial q_i \partial \bar{q}_j}) \xrightarrow{w} \det(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}),$$

where the convergence of measures in understood in the weak sense.

3.5 Remark. (1) It is easy to see that if $u_N \to u$ uniformly on compact subsets, and $u_N \in C(\Omega) \cap P(\Omega)$ then $u \in C(\Omega) \cap P(\Omega)$.

(2) Note that the real analogue of this result was proved by A.D. Aleksandrov [2], and the complex analogue by Chern, Levine, and Nirenberg [19].

We will need a refinement of Theorem 3.4 which was proved by the author in [7] in somewhat different notation. We denote for brevity $\partial^2 u := \left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right)$.

3.6 Theorem ([7]). *Let $\Omega \subset \mathbb{H}^n$ be an open subset. Fix $k = 1, \dots, n$. Let $\{u_N^{(i)}\}_{N=1}^{\infty}$, $1 \leq i \leq k$, be sequences in $P(\Omega) \cap C(\Omega)$. Let $V^{(1)}, \dots, V^{(n-k)}$ be continuous functions*

on Ω with values in the space of $(n \times n)$-hyperhermitian matrices. Assume that for every $i = 1, \dots, k$

$$u_N^{(i)} \to u^{(i)} \text{ as } N \to \infty$$

uniformly on compact subsets. Then $u^{(i)} \in P(\Omega) \cap C(\Omega)$, and

$$\det\left(\partial^2 u_N^{(1)}, \dots, \partial^2 u_N^{(k)}, V^{(1)}, \dots, V^{(n-k)}\right) \xrightarrow{w} \det\left(\partial^2 u^{(1)}, \dots, \partial^2 u^{(k)}, V^{(1)}, \dots, V^{(n-k)}\right)$$

weakly in the sense of measures (where we used the notion of mixed determinant).

For hyperhermitian matrices $A, B_1, \dots, B_{n-k}$ let us denote

$$\det(A[k], B_1, \dots, B_{n-k}) := \det(\underbrace{A, \dots, A}_{k \text{ times}}, B_1, \dots, B_{n-k}).$$

Note that the maximum of two plurisubharmonic functions is again plurisubharmonic.

3.7 Theorem ([7]). *Let $\Omega \subset \mathbb{H}^n$ be an open subset. Fix $k = 1, \dots, n$. Let $f, g \in P(\Omega) \cap C(\Omega)$. Assume that $\min\{f, g\} \in P(\Omega) \cap C(\Omega)$. Let $V^{(1)}, \dots, V^{(n-k)}$ be continuous functions on Ω with values in the space of $(n \times n)$-hyperhermitian matrices. Then*

$$\begin{aligned} \det(\partial^2(\max\{f,g\})[k], V^{(1)}, \dots, V^{(n-k)}) = \\ \det(\partial^2 f[k], V^{(1)}, \dots, V^{(n-k)}) + \det(\partial^2 g[k], V^{(1)}, \dots, V^{(n-k)}) - \\ \det(\partial^2(\min\{f,g\})[k], V^{(1)}, \dots, V^{(n-k)}). \end{aligned}$$

3.8 Remark. Theorem 3.7 was proved in [7] as a consequence of more precise result, Theorem 3.2.1 there, which is a quaternionic version of a result by Błocki [13] for complex plurisubharmonic functions.

4 Applications to valuation theory.

Let us discuss the applications of the above results to the theory of valuations on convex sets. The necessary definitions of this theory were reminded in Introduction. Here we would like to recall Theorem 0.3.

4.1 Theorem ([7], Theorem 4.1.3). *Fix $k = 1, \dots, n$. Fix $\psi \in C_0(\mathbb{C}^n, \Omega^{n-k,n-k})$. Then $K \mapsto \int_{\mathbb{C}^n} (dd^c h_K)^k \wedge \psi$ defines a continuous translation invariant valuation on $\mathcal{K}(\mathbb{C}^n)$.*

This result in the above generality was proved by the author in [7] using some known properties of complex plurisubharmonic functions. This theorem contains two non-trivial parts: the continuity and the valuation property of the above functional. The continuity is a consequence of the Chern-Levine-Nirenberg theorem [19] and the fact that a sequence $\{K_N\}$ of convex compact sets converges in the Hausdorff metric to a convex compact set K if and only if $h_{K_N} \to h_K$ uniformly on compact subsets. The valuation property is a consequence of the Błocki formula [13] combined with the facts that

$$\begin{aligned} h_{K_1 \cup K_2} = \max\{h_{K_1}, h_{K_2}\} \text{ when } K_1 \cup K_2 \text{ is convex, and} \\ h_{K_1 \cap K_2} = \min\{h_{K_1}, h_{K_2}\}. \end{aligned}$$

Let us discuss the quaternionic version of Theorem 4.1. In order to simplify the exposition we will state the result in a simple minded form. A better way was discussed in [7] where quaternionic analogues of (p,p)-forms were introduced.

4.2 Theorem ([7], Theorem 4.2.1). *Fix $k = 1, \dots, n$. Let ψ_0 be a continuous compactly supported real valued function on $\mathbb{H}^{n*}$. Let $V^{(1)}, \dots, V^{(n-k)}$ be continuous compactly supported functions on $\mathbb{H}^{n*}$ with values in the space of $(n \times n)$-hyperhermitian matrices. Then the functional*

$$K \mapsto \int_{\mathbb{H}^{n*}} \det(\partial^2 h_K[k], V^{(1)}, \dots, V^{(n-k)}) \cdot \psi_0 \cdot dvol$$

is a translation invariant continuous valuation.

Similarly to the complex case, the continuity is a consequence of Theorem 3.6, and the valuation property is a consequence of Theorem 3.7.

5 Quaternionic Monge-Ampère equations.

In this section we will discuss some results on quaternionic Monge-Ampère equations following [5]. They have real and complex analogues; the references will be given below.

5.1 Definition. An open bounded domain $\Omega \subset \mathbb{H}^n$ with a smooth boundary $\partial\Omega$ is called strictly pseudoconvex if for every point $z_0 \in \partial\Omega$ there exists a neighborhood $\mathcal{O}$ and a smooth quaternionic strictly plurisubharmonic function h on $\mathcal{O}$ such that $\Omega \cap \mathcal{O} = \{h < 0\}$, $h(z_0) = 0$, and $\nabla h(z_0) \neq 0$.

Let B denote the unit Euclidean ball in $\mathbb{H}^n$.

5.2 Theorem ([5], Theorem 0.1.4). *Let $f \in C^\infty(\bar{B})$, $f > 0$. Let $\phi \in C^\infty(\partial B)$. There exists unique function $u \in C^\infty(\bar{B})$ which is quaternionic plurisubharmonic in B and which is a solution of the Dirichlet problem*

$$\det\left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right) = f \text{ in } B,$$

$$u|_{\partial B} = \phi.$$

5.3 Remark. (1) It is natural to expect that Theorem 5.2 is true for a larger class of domains, say for all bounded strictly pseudoconvex domains.

(2)The real version of Theorem 5.2 was proved for arbitrary bounded strictly convex domains in $\mathbb{R}^n$ by Caffarelli, Nirenberg, and Spruck [15]. The complex version of it was proved for arbitrary bounded strictly pseudoconvex domains in $\mathbb{C}^n$ by Caffarelli, Kohn, Nirenberg, and Spruck [16] and Krylov [29]. Our method is a modification of the method of the paper [16]. Note also that in the case $n = 1$ the problem is reduced to the classical Dirichlet problem for the Laplacian in $\mathbb{R}^4$ (which is a linear problem); it was solved in XIX century. Note also that *interior* regularity of the solution of the Dirichlet problem for real Monge-Ampère equations was proved earlier by A. Pogorelov, and the proof was briefly described in [34]-[36]. The complete proof was published in [37] and [17], [18].

5.4 Theorem ([5], Theorem 0.1.3). *Let $\Omega \subset \mathbb{H}^n$ be a bounded quaternionic strictly pseudoconvex domain. Let $f \in C(\bar{\Omega})$, $f \geq 0$. Let $\phi \in C(\partial\Omega)$. Then there exists unique function $u \in C(\bar{\Omega})$ which is plurisubharmonic in Ω and such that*

$$\det\left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right) = f \text{ in } \Omega,$$

$$u|_{\partial\Omega} \equiv \phi.$$

5.5 Remark. The real analogue of this result was proved by A.D. Aleksandrov [2], and the complex one by E. Bedford and B.A. Taylor [12].

6 Generalizations to hypercomplex manifolds.

Some of the definitions and results on the quaternionic plurisubharmonic functions discussed above were extended by M. Verbitsky and the author [8] to a more general context of so called hypercomplex manifolds. In this section we give an overview of these results including a geometric interpretation of quaternionic strictly plurisubharmonic functions as (local) potentials of HKT-metrics. The exposition follows [8].

6.1 Definition. A *hypercomplex* manifold is a smooth manifold X together with a triple (I, J, K) of complex structures satisfying the usual quaternionic relations:

$$IJ = -JI = K.$$

6.2 Remark. (1) We suppose here (in the opposite to much of the literature on the subject) that the complex structures I, J, K act on the *right* on the tangent bundle TX of X. This action extends uniquely to the right action of the algebra $\mathbb{H}$ of quaternions on TX.

(2) It follows that the dimension of a hypercomplex manifold X is divisible by 4.

Let (X^{4n}, I, J, K) be a hypercomplex manifold. Let us denote by $\Lambda_I^{p,q}(X)$ the vector bundle of (p,q)-forms on the complex manifold (X, I). By the abuse of notation we will also denote by the same symbol $\Lambda_I^{p,q}(X)$ the space of C^∞-sections of this bundle.

Let

$$\partial\colon \Lambda_I^{p,q}(X) \to \Lambda_I^{p+1,q}(X) \tag{2}$$

be the usual ∂-differential of differential forms on the complex manifold (X, I).

Set

$$\partial_J := J^{-1} \circ \bar{\partial} \circ J. \tag{3}$$

6.3 Claim ([39]). *(1)$J\colon \Lambda_I^{p,q}(X) \to \Lambda_I^{q,p}(X)$.*

(2) $\partial_J\colon \Lambda_I^{p,q}(X) \to \Lambda_I^{p+1,q}(X)$.

(3) $\partial\partial_J = -\partial_J\partial$.

6.4 Definition ([39]). Let $k = 0, 1, \dots, n$. A form $\omega \in \Lambda_I^{2k,0}(X)$ is called *real if*

$$\overline{J \circ \omega} = \omega.$$

We will denote the subspace of real C^∞-smooth $(2k,0)$-forms on (X,I) by $\Lambda^{2k,0}_{I,\mathbb{R}}(X)$.

6.5 Lemma. *Let X be a hypercomplex manifold. Let $f\colon X\to\mathbb{R}$ be a smooth function. Then $\partial\partial_J f\in\Lambda^{2,0}_{I,\mathbb{R}}(X)$.*

6.6 Definition. Let $\omega\in\Lambda^{2,0}_{I,\mathbb{R}}(X)$. Let us say that ω is non-negative (notation: $\omega\geq 0$) if

$$\omega(Y,Y\circ J)\geq 0$$

for any (real) vector field Y on the manifold X. Equivalently, ω is non-negative if $\omega(Z,\bar{Z}\circ J)\geq 0$ for any $(1,0)$-vector field Z.

6.7 Definition. A continuous function

$$h: X\to\mathbb{R}$$

is called quaternionic plurisubharmonic if $\partial\partial_J h$ is a non-negative (generalized) section of $\Lambda^{2,0}_{I,\mathbb{R}}(X)$.

6.8 Remark. The non-negativity in the generalized sense is discussed in detail in [8], Section 5.

Let us denote by $P'(X)$ the class of continuous quaternionic plurisubharmonic functions on X. Let us denote by $P''(X)$ the subclass of functions from $P'(X)$ with the following additional property: a function $h\in P'(X)$ belongs to $P''(X)$ if and only if any point $x\in X$ has a neighborhood $U\ni x$ and a sequence $\{h_N\}\subset P'(U)\cap C^2(U)$ such that $h_N\to h$ uniformly on compact subsets of U. Thus $P''(X)\subset P'(X)$.

We conjecture that $P'(X)=P''(X)$. This conjecture is true when X is an open subset of $\mathbb{H}^n$.

6.9 Theorem ([8], Theorem 1.10). *Let X be a hypercomplex manifold of (real) dimension $4n$. Let $0<k\leq n$. For any $h^{(1)},\dots,h^{(k)}\in P''(X)$ one can define a non-negative generalized section of $\Lambda^{2k}_{I,\mathbb{R}}$ denoted by $\partial\partial_J h^{(1)}\wedge\dots\wedge\partial\partial_J h^{(k)}$ which is uniquely characterized by the following two properties:*

(1) if $h^{(1)},\dots,h^{(k)}\in C^2(X)$ then the definition is clear;

(2) if $\{h^{(i)}_N\}\subset P''(X)$, $h^{(i)}_N\to h^{(i)}$ as $N\to\infty$ uniformly on compact subsets , $i=1,\dots,k$, then $h^{(i)}\in P''(X)$ and

$$\partial\partial_J h^{(1)}_N\wedge\dots\wedge\partial\partial_J h^{(k)}_N\to\partial\partial_J h^{(1)}\wedge\dots\wedge\partial\partial_J h^{(k)}$$

in the weak topology on measures.

6.10 Remark. Theorem 6.9 generalizes Theorem 3.6 from the flat space $\mathbb{H}^n$ to hypercomplex manifolds.

Let us discuss the relations to the HKT-geometry. Let g be a Riemannian metric on a hypercomplex manifold X. The metric g is called *quaternionic Hermitian* (or hyperhermitian) if g is invariant with respect to the group $SU(2)\subset\mathbb{H}$ of unitary quaternions.

Given a quaternionic Hermitian metric g on a hypercomplex manifold X, consider the differential form

$$\Omega := \omega_J - \sqrt{-1}\omega_K$$

where $\omega_L(A, B) := g(A, B \circ L)$ for any $L \in \mathbb{H}$ with $L^2 = -1$ and any real vector fields A, B on X. It is easy to see that Ω is a $(2, 0)$-form with respect to the complex structure I.

6.11 Definition. The metric g on X is called HKT-metric if

$$\partial\Omega = 0.$$

6.12 Remark. HKT-metrics on hypercomplex manifolds first were introduced by Howe and Papadopoulos [25]. Their original definition was different but equivalent to Definition 6.11 (see [22]).

Let us denote by $S_{\mathbb{H}}(X)$ the vector bundle over X such that its fiber over a point $x \in X$ is equal to the space of hyperhermitian forms on the tangent space T_xX. Consider the map of vector bundles

$$t \colon \Lambda^{2,0}_{I,\mathbb{R}}(X) \to S_{\mathbb{H}}(X)$$

defined by $t(\eta)(A, A) = \eta(A, A \circ J)$ for any (real) vector field A on X. Then t is an isomorphism of vector bundles (this was proved in [39]).

6.13 Theorem ([8], Prop. 1.14). *(1) Let f be an infinitely smooth strictly plurisubharmonic function on a hypercomplex manifold (X, I, J, K). Then $t(\partial\partial_J f)$ is an HKT-metric.*

(2) Conversely assume that g is an HKT-metric. Then any point $x \in X$ has a neighborhood U and an infinitely smooth strictly plurisubharmonic function f on U such that $g = t(\partial\partial_J f)$ in U.

6.14 Remark. (i) On the flat space $\mathbb{H}^n$ one has for any smooth real valued function f

$$t(\partial\partial_J f) = \frac{1}{4}\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right)$$

by Proposition 4.1 of [8].

(ii) The proof of Theorem 6.13 uses a result of Banos-Swann [11].

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